Bemerkung

Bosonen $[\Psi(x), \Psi^+(x')] = \delta(x-x')$

Fermionen $\{\Psi(x), \Psi^+(x')\} = \delta(x-x')$ $\qquad \Psi(x) = \sum_i \varphi_i(x) a_i$

Das Bose-Gas (Verdünnt) mit schwacher Wechselwirkung

N. Bogolyubov (1947) $\qquad T=0$

$\varepsilon_p = \frac{p^2}{2m} \quad \dots \quad \overset{WW}{\Longrightarrow} \quad \varepsilon_p = cp$ superfluid

$$H = \sum_k \varepsilon(k) a_k^+ a_k + \frac{1}{2V} \sum_{kk'q} U(q) a_{k+q}^+ a_{k'-q}^+ a_{k'} a_k \qquad \boxed{\varepsilon(k) = \frac{\hbar^2 k^2}{2m}}$$

ohne WW: $U(q) = 0 \qquad a_0 = a_{k=0};\ a_0^+ = a_{k=0}^+$

$$\boxed{\begin{array}{l} U(q) = \int d^3r\, U(r) e^{-iqr} \\ U(q) = \text{const} \Rightarrow U(r) \propto \delta(r) \end{array}}$$

$a_0^+ a_0 |\phi_0\rangle = N|\phi_0\rangle$ Grundzustand ϕ_0

$a_k^+ a_k |\phi_0\rangle = 0 \qquad k \neq 0$

mit schwacher WW: $a_0^+ a_0 |\phi\rangle = N_0 |\phi\rangle \qquad N_0 < N \qquad \frac{N_0}{V} = n_0 > 0$

$a|N\rangle = \sqrt{N}|N\rangle \qquad a$ ist „groß"

$$\underbrace{a_0 a_0^+}_{N_0+1} - \underbrace{a_0^+ a_0}_{N_0} = 1 \quad \Longrightarrow \quad a_0 = \sqrt{N_0}\, e^{i\varphi_0};\ a_0^+ = \sqrt{N_0}\, e^{-i\varphi_0}$$

wähle $\varphi_0 = 0$

$\langle \phi_N | a_0 | \phi_{N+1} \rangle = \sqrt{N_0}$

$$H = \sum_k \varepsilon(k) a_k^+ a_k + \frac{U}{2V} N_0^2 + \frac{U}{2V} \sum_{q\neq 0} N_0 \left(a_q^+ a_{-q}^+ + a_{-q} a_q + 4 a_q^+ a_q\right) + \dots$$

Gesamtzahl der Teilchen $N = N_0 + \sum_{q\neq 0} a_q^+ a_q \Rightarrow N_0 = N - \sum_{q\neq 0} a_q^+ a_q$

$$H = H_{kin} + \frac{U}{2V}\left(N - \sum_{q\neq 0} a_q^+ a_q\right)^2 + \frac{U}{2V}\left(N - \sum_{q\neq 0} a_q^+ a_q\right) \sum_{k\neq 0}\left(a_k^+ a_{-k}^+ + a_{-k} a_k + 4 a_k^+ a_k\right)$$

$$= H_{kin} + \frac{U}{2V}\left(N^2 + 2N \sum_{k\neq 0} a_k^+ a_k + N \sum_{k\neq 0}\left(a_k^+ a_{-k}^+ + a_{-k} a_k\right)\right) + \dots$$

Dichte $n = \frac{N}{V}$

$$\boxed{H = H_{kin} + Un\frac{N}{2} + Un \sum_{k\neq 0} a_k^+ a_k + \frac{Un}{2} \sum_{k\neq 0}\left(a_k^+ a_{-k}^+ + a_{-k} a_k\right)}$$

$\Big(H = \sum_k \varepsilon(k) a_k^+ a_k$ ist sofort lösbar

mit WW

$\Longrightarrow H = \sum_k \left(\varepsilon(k) + Un\right) a_k^+ a_k + \text{const.} \Big\}$ hier nicht der Fall $\Big)$

- **Ziel** ist es den Hamiltonoperator auf die Form zubringen:

$$H = \sum_k E(k)\, \gamma_k^+ \gamma_k + \text{const} \quad (\text{diagonalisierbar})$$

- **Transformation** von Bogl.

$$a_k = \cosh(\Theta)\gamma_k - \sinh(\Theta)\gamma_{-k}^+ \qquad \Theta = \Theta(k) \qquad \cosh(\Theta) = ch$$

$$a_k^+ = -\sinh(\Theta)\gamma_{-k} + \cosh(\Theta)\gamma_k^+ \qquad \Theta(-k) = \Theta(k) \qquad \sinh(\Theta) = sh$$

$$\boxed{\begin{array}{l} [\gamma_k, \gamma_{k'}^+] = \delta_{kk'} \\ [\gamma_k\, \gamma_{k'}] = 0 \\ [\gamma_{k'}^+ \gamma_{k'}^+] = 0 \end{array}}$$

$[a_k, a_k] = 0$ für $k' \neq k$, $k' \neq -k$

$$= ch^2[\gamma_k \gamma_k] + sh^2[\gamma_{-k}^+ \gamma_{-k}^+]$$

$$- sh\, ch\, [\gamma_k\, \gamma_{-k}^+]\; sh\, ch\, [\gamma_{-k}^+ \gamma_k] = 0$$

$$[a_k, a_k^+] = -ch\, sh\, \underbrace{[\gamma_k\, \gamma_{-k}]}_{0} + ch^2 \underbrace{[\gamma_k \gamma_k^+]}_{1} + sh^2 \underbrace{[\gamma_{-k}^+ \gamma_{-k}]}_{-1} - sh\, ch\, \underbrace{[\gamma_{-k}^+ \gamma_k^+]}_{0} = 1$$

$$H = \sum_k \varepsilon(k)\,(-sh\,\gamma_{-k} + ch\,\gamma_k^+)(ch\,\gamma_k - sh\,\gamma_{-k}^+) + Un\frac{N}{2}$$

$$= \sum_k (\varepsilon_k + Un)(-sh\,\gamma_{-k} + ch\,\gamma_k^+)(ch\,\gamma_k - sh\,\gamma_{-k}^+)$$

$$+ U\frac{n}{2}\sum_{k\neq 0}\Big((-sh\,\gamma_{-k} + ch\,\gamma_k^+)(-sh\,\gamma_k + ch\,\gamma_{-k}^+) + (ch\,\gamma_{-k} - sh\,\gamma_k^+)(ch\,\gamma_k - sh\,\gamma_{-k}^+)\Big)$$

$$\boxed{\begin{array}{l} 2\, sh\, ch = \sinh(2\Theta) \\ sh^2 + ch^2 = \cosh(2\Theta) \end{array}}$$

- Terme mit $\gamma_k \gamma_k$

$$\sum_k (\varepsilon(k) + nU)(-sh\, ch) + U\frac{n}{2}\sum_k (sh^2 + ch^2) = 0$$

$$\tan(2\Theta) = \frac{Un}{\varepsilon(k) + Un}$$

$$\gamma_k \gamma_k^+ = 1 + \gamma_k^+ \gamma_k$$

$$\Rightarrow \boxed{E(k) = \sqrt{(\varepsilon(k) + nU)^2 - (nU)^2}} \qquad \boxed{\varepsilon(k) = \frac{\hbar^2 k^2}{2m}}$$

- γ_k^+ Erzeugt Quasiteilchen mit Impuls $\hbar k$
- $T = 0$ "$\gamma_k^+ \gamma_k = 0$" für $\gamma_k^+ \gamma_k \,|\phi_0\rangle = 0$

  $T > 0$ $\gamma_k^+ \gamma_k > 0$
- $\varepsilon(k) \ll Un$: $E(k) = \sqrt{2Un\,\varepsilon + \cancel{\varepsilon^2}} = \sqrt{2Un\frac{\hbar^2 k^2}{2m}} = c|k|$

  Schallgeschwindigkeit $c = \sqrt{2\frac{Un\,\hbar^2}{m}}$